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Algebraic Thinking in the Early Grades: What Is It?[1]

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Abstract: This reaction paper is written from the perspective of a researcher whose work in the learning of algebra has primarily been with 12- to 16-year-olds. To provide some context for the comments and questions I will be raising with respect to the set of papers contained in this collection, I begin with remarks of a theoretical nature, first focusing on distinctions between arithmetic and algebraic thinking, and then elaborating the main components of algebraic activity. Against the backdrop of these theoretical remarks, I discuss some of the differences and similarities among the described curricula regarding the development of algebraic thinking. Lastly, I offer a definition for algebraic thinking in the early grades that is integrated within an existing model of algebraic activity in the later grades.

Introduction

From the time of Al-Khowarizmi and his fellow Arab mathematicians in the 9th century, algebra has been viewed as the science of equation solving. Some eleven centuries later, while that view had not changed to any great extent, the age of the students studying algebra had. Latter 20th century students, not having to wait until reaching adulthood to begin their labor with the literal symbol and its manipulation, usually advanced to the study of algebra sometime during high school, after they had completed the study of arithmetic. However, for many, algebra learning was not the labor of love that it had been for Al-Khowarizmi and his fellow mathematicians. Research conducted during the 1970s and 80s pointed to some of the difficulties that students were encountering with this mathematical subject. Some reform-oriented scholars in the late 1980s then conjectured that, if we were to rethink what is central to the core of algebra and were to introduce certain elements earlier – within the elementary school program of mathematics – perhaps algebra could become accessible to a larger majority of students. This paper touches upon some of the current issues surrounding that effort by means of examples provided by the curricula described in this volume, in particular, their approaches to the development of algebraic thinking in the earlier grades.

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Some Theoretical Considerations

Arithmetic thinking versus algebraic thinking

The recent book, *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) features as one of its chapters the growth of mathematical proficiency beyond number. That chapter includes discussion of the kinds of thinking that students tend to develop in traditional arithmetic programs as distinguished from those that are required for the study of algebra:

> In the transition from arithmetic to algebra, students need to make many adjustments, even those students who are quite proficient in arithmetic. At present, for example, elementary school arithmetic tends to be heavily answer-oriented and does not focus on the representation of relations. Students beginning algebra, for whom a sum such as 8 + 5 is a signal to compute, will typically want to evaluate it and then, for example, write 13 for the box in the equation 8 + 5 = – + 9 instead of the correct value of 4. When an equal sign is present, they treat it as a separator between the problem and the solution, taking it as a signal to write the result of performing the operations indicated to the left of the sign. Or, when doing a sequence of computations, students often treat the equal sign as a left-to-right directional signal. … Students oriented toward computation are also perplexed by an expression such as $x + 3$; they think they should be able to do something with it, but are unsure as to what that might be. They are not disposed to think about the expression itself as being the subject of attention. Similarly, they need to rethink their approach to problems. In solving a problem such as "When 3 is added to 5 times a certain number, the sum is 38; find the number," students emerging from arithmetic will subtract 3 from 38 and then divide by 5 – undoing in reverse order, as they have been taught, the operations stated in the problem text. In contrast, they will be taught in algebra classes first to represent the relationships in the situation by using the stated operations: $5x + 3 = 38$. (pp. 261-262)

As the above suggests, students operating in an arithmetic frame of reference tend not to see the relational aspects of operations; their focus is on calculating. Thus, considerable adjustment is required in developing an algebraic way of thinking, which includes, but is not restricted to:

1. A focus on relations and not merely on the calculation of a numerical answer;
2. A focus on operations as well as their inverses, and on the related idea of doing / undoing;
3. A focus on both representing and solving a problem rather than on merely solving it;

4. A focus on both numbers and letters, rather than on numbers alone. This includes:
   (i) working with letters that may at times be unknowns, variables, or parameters;
   (ii) accepting unclosed literal expressions as responses;
   (iii) comparing expressions for equivalence based on properties rather than on numerical evaluation;
5. A refocusing of the meaning of the equal sign.

The research (see, e.g., Wagner & Kieran, 1989; Kieran, 1992; Bednarz, Kieran, & Lee, 1996) that has centered on the difficulties in moving from an arithmetic to an algebraic form of reasoning has, by extension, provided a basis for some of the changes to be found in current arithmetic programs of study – changes that encourage the emergence of algebraic thinking in the earlier grades. However, one needs to look more widely than this body of research in order to have a fuller account of the influences that have come to bear on such curricula. These influences include the work of mathematics educators and researchers who have offered alternative ways of conceptualizing the area of school algebra, as well as the initiatives of the National Council of Teachers of Mathematics – more specifically in its Standards documents and other related publications (NCTM, 1989, 1998, 2000).

A model of algebraic activity

A number of different characterizations of algebra can be found in the mathematics education literature. For example, Usiskin (1988) described four conceptions of algebra: generalized arithmetic, the set of procedures used for solving certain problems, the study of relationships among quantities, and the study of structures. Kaput (1995) identified five aspects of algebra: generalization and formalization; syntactically guided manipulations; the study of structure; the study of functions, relations, and joint variation; and a modeling language. A discussion document published by the National Council of Teachers of Mathematics (1998, see Appendix E) describes four organizing themes for school algebra: functions and relations, modeling, structure, and language and representation. Kieran (1996) categorized school algebra according to the activities typically engaged in by students: generational activities, transformational activities, and global meta-level activities. We shall look more closely at this latter categorization as it will serve as a basis for offering in the last section of this paper an integrated perspective on, and a new definition for, algebraic thinking.

According to the Kieran (1996) model, the generational activities of algebra involve the forming of the expressions and equations that are the objects of algebra. Typical examples include: i) equations containing an unknown that represent problem situations (see, e.g., Bell, 1995), ii) expressions of generality arising from geometric patterns or numerical sequences (see, e.g., Mason, 1996), and iii) expressions of the rules governing numerical relationships (see, e.g., Lee & Wheeler, 1987). The underlying objects of expressions and equations are variables and unknowns, and so these too are included in the generational activity of algebra, as are the equal sign and the notion of equation solution. Much of the meaning-building for algebraic objects occurs within the generational activity of algebra.

The second type of algebraic activity – the transformational ("rule-based") activities – includes, for instance, collecting like terms, factoring, expanding, substituting, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations, simplifying expressions, working with equivalent expressions and equations, and so on. A great deal of this type of activity is concerned with changing the form of an expression or equation in order to maintain equivalence.

Lastly, there are the global, meta-level, mathematical activities. These are the activities for which algebra is used as a tool but which are not exclusive to algebra. They include problem solving, modeling, noticing structure, studying change, generalizing, analyzing relationships, justifying, proving, and predicting – activities that could be engaged in without using any algebra at all. In fact, they suggest more general mathematical processes and activity. However, attempting to divorce these meta-level activities from algebra removes any context or need that one might have for using algebra. Indeed, the global meta-level activities are essential to the other activities of algebra, in particular, to the meaning-building generational activities; otherwise all sense of purpose is lost.

In the model of algebraic activity described above, and which was presented at the ICME-8 Congress in Seville in 1996, the view that was expressed implicitly involved the letter-symbolic representation. Up until very recently, we have privileged the letter-symbolic in algebraic representations. But the advent of computing technology, with its facilitating of other means of representing relationships and of operating on these relationships, in ways analogous to the generational and transformational activities of algebra, led to my proposing in 1996 a definition for algebraic thinking that did not involve necessarily the letter-symbolic (the term "algebraic thinking" being used to distinguish it from traditional school algebra):

> Algebraic thinking can be interpreted as an approach to quantitative situations that emphasizes the general relational aspects with tools that are not necessarily letter-symbolic, but which can ultimately be used as cognitive

> support for introducing and for sustaining the more traditional discourse of school algebra. (Kieran, 1996, p. 275)

However, the approach to defining algebraic thinking that I offer later in this paper is somewhat broader than the above in that it is not prompted by the contributions made possible by technology, but rather by the directions that have recently been taken in current curricular efforts in the early grades. But, first, we consider another thread that has been woven into the fiber of algebra in certain countries – namely, the thread of functional approaches.

Functional approaches

Towards the end of the 1980s, and concurrent with early research work involving the use of technology in the learning of algebra, functional approaches began to permeate algebraic activity. Heid (1996) defined these approaches as follows:

> The functional approach to the emergence of algebraic thinking … suggests a study of algebra that centers on developing experiences with functions and families of functions through encounters with real world situations whose quantitative relationships can be described by those models. (p. 239)

Despite disagreement by some (see, e.g., Lee, 1997) as to whether functions are really a part of algebra, a functional view of expressions and equations, and of algebra in general, is quite widespread in the U.S. mathematics education community (e.g., Fey & Heid, 1991; Schwartz & Yerushalmy, 1992; Romberg, Fennema, & Carpenter, 1993; Smith, 2003) and is also reflected in the *Principles and Standards for School Mathematics*, published by the National Council of Teachers of Mathematics (2000):

> The Algebra Standard emphasizes relationships among quantities, including functions, ways of representing mathematical relationships, and the analysis of change. Functional relationships can be expressed by using symbolic notation, which allows complex mathematical ideas to be expressed succinctly and change to be analyzed efficiently. … By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more-sophisticated work in algebra in the middle grades and high school. For example, systematic experience with patterns can build up to an understanding of the idea of function, and experience with numbers and their properties lays a foundation for later work with symbols and algebraic expressions. (p. 37)

While the functional thread clearly runs through the Algebra Standard in each of the four grade bands (K-2, 3-5, 6-8, 9-12), the goals of the algebra standard are somewhat more wide-ranging than those associated uniquely with a functional approach. The grades 3-5 band, for example, states that:

> In grades 3-5, algebraic ideas should emerge and be investigated as students: (a) identify or build numerical and geometric patterns; (b) describe patterns verbally and represent them with tables or symbols; (c) look for and apply relationships between varying quantities to make predictions; (d) make and explain generalizations that seem to always work in particular situations; (e) use graphs to describe patterns and make predictions; (f) explore number properties; and (g) use invented notation, standard symbols, and variables to express a pattern, generalization, or situation. (NCTM, 2000, p. 159)

Thus, one notes in the standard for this grade band a focus on "algebraic ideas" that encompass not only patterns, relationships, generalizations, and their representation with different kinds of symbols, but also number properties and their exploration (item f). However, the latter tend to be somewhat eclipsed in the stated goals by the attention given to patterns, their generalization, and their representation.

The above sample suggests a somewhat eclectic situation regarding views of algebra and algebraic thinking. Currently, there is no one vision or perspective that has been adopted at large in the international community of mathematics educators with respect to what is meant by algebraic thinking in the early grades. With this diversity of points of view as a backdrop, I now look briefly at the collected papers of this special issue with the double aim of drawing out the positions of each with respect to algebraic thinking and of leading towards a framework for thinking about "algebraic thinking in the early grades" – a framework that is both grounded in actuality and makes contact with existing "frameworks for thinking about algebra in the later grades."

Differences and Similarities among the Papers in this Collection

Readers will likely have found more differences than similarities among the various papers of the collection. Perhaps the most striking difference concerns the moment in the curriculum when symbolic algebra is introduced – from the teaching of algebra with literal symbols in the Davydov curriculum right from the first grade, even preceding the study of arithmetic, to the nearly-complete absence of symbolic algebra in the K-5 *Investigations* curriculum.

Another seeming point of difference in the various curricula centers on the definition or view of that which constitutes algebra and/or algebraic thinking. For example, the Davydov curriculum is said to:

> Develop children's ability to think in a variety of ways that foster algebraic performance. First, it develops theoretical thinking, which according to Vygotsky comprises the essence of algebra. For example, the children develop a habit of searching out relationships among quantities across contextualized situations, and learn to solve an equation by attending to its underlying structure. The curriculum develops children's capacity for analysis and generalization. Their ability to interpret a letter as "any number" allows the teacher to introduce children to the kind of general argument that is the hallmark of algebraic justification and proof. (Schmittau & Morris, p. 23) [underlining added, in this and all subsequent quotations]

However if, instead of focusing on the obvious differences among the curricula, we look for characteristics that are held in common, then certain themes begin to emerge. For instance, the Chinese curriculum has as its overarching goal of learning algebra, "to better represent and understand quantitative relationships," in addition to the focus on "equations and equation solving" (students see equations with placeholders throughout first grade). Cai states that, "the concepts of variables and functions are not formally defined in Chinese elementary school mathematics"; however, in his view, the program of study "is designed to permeate variable and function ideas throughout the curriculum to develop students' function sense" (p. 4). According to Cai, from the first grade onward, students experience "problem solving that involves the comparison of several quantitative relationships" (p. 7), "the pull toward generalization" (p. 14), and "mathematical modeling of problem situations" (p. 10).

The Singapore primary mathematics curriculum does not treat algebra explicitly until the formal introduction at primary six, the final year of primary school: "At this level, the emphases are on the developing of algebraic concepts and algebraic manipulation skills … to use letters to represent unknown numbers, to write simple algebraic expressions in one variable involving one operation, … to find the value of a simple algebraic expression in one variable by substitution, to simplify algebraic expressions in one variable involving addition and subtraction, and to solve word problems involving algebraic expressions" (Ng, p. 40). However, Ng argues that activities that foster algebraic thinking are included in the primary mathematics curriculum even if they are not listed overtly as such (algebraic thinking is not mentioned explicitly in the Singaporean curriculum). She states that the three approaches that develop algebraic thinking in this curriculum are the "generalization approach, the functional approach, and the problem-solving approach" (p. 3). The thinking processes that are said to support these three approaches are: generalizing and specializing, doing and undoing, and analyzing parts and whole. The curriculum, which shares, in spirit, many of the same goals as

NCTM's *Principles and Standards*, proposes activities aimed at "understanding patterns, relations, and functions" "problem solving involving relational perspectives" (p. 6), and developing functional interpretations" (p. 3).

The Korean mathematics curriculum, which is said to emphasize symbols in the formal algebra course that begins in the 7^{th} grade, defines algebra as follows:

> Algebra is a subject dealing with expressions with symbols and the extended numbers beyond the whole numbers in order to solve equations, to analyze functional relations, and to determine the structure of the representational system, which consists of expressions and relations. However, activities such as solving equations, analyzing functional relations and determining structure are not the purpose of algebra, but tools for modeling of real world phenomena and problem solving related to the various situations. Furthermore, algebra is much more than the set of knowledge and techniques. It is a way of thinking. Success in algebra depends on at least six kinds of mathematical thinking abilities as follows: generalization, abstraction, analytic thinking, dynamic thinking, modeling, and organization. (Lew, pp. 5-6)

In the Korean curriculum, the development of algebraic thinking at the elementary school level is based on the elaboration of activities related to these latter six kinds of mathematical thinking.

The U.S. *Investigations in Number, Data, and Space* K-5 curriculum identifies "the mathematics of change" as the central unifying theme for the study of algebra: "Work in the *Investigations* curriculum emphasizes qualitative understanding of these ideas of mathematical change as students discuss the meaning of graphical and numerical patterns" (Russell et al., cited in Moyer, Huinker, & Cai, p. 11). The mathematics of change also serves as "the impetus behind the development of the big ideas of patterns and relationships, representation, and modeling" (p. 12). It is noted that the fifth grade unit on patterns of change is the capstone unit in the algebra strand and provides experiences in describing, representing and comparing rates of change; however, it is emphasized that, "all students are expected to come up with a general rule, but not necessarily an algebraic equation" (p. 27) and that, "it is not the intent of the curriculum that students develop the ability to formally represent functions with algebraic symbols" (p. 31).

The emphasis on patterns, which is found in the *Investigations* curricular approach to algebra and which is consistent with NCTM's *Principles and Standards*, is contrasted with important elements of the Davydov curriculum (Schmittau & Morris):

> The Algebra Standard [in NCTM's *Principles and Standards for School Mathematics*] stresses the child's discovery of relationships, mathematical generalizations, and patterns involving numbers and geometric objects. The inductive discovery aspect of the Algebra Standard is absent in the Russian curriculum, nor is there any work with patterns. … Nor does the Russian curriculum use experience with number as the basis for developing algebra, but instead uses relationships between quantities as the foundation. For example, it does not teach children to solve equations by thinking about "doing and undoing" numerical operations, but instead teaches them to solve equations by thinking of them in terms of relationships between quantities. … These differences [and others] reflect fundamental differences in the bases for developing algebraic understandings, and divergent suppositions about the precursors of algebraic thought. (p. 84)

Notwithstanding these foundational differences between the Davydov curriculum and the other four that are presented in this special issue, there are clearly some points of contact to be found with respect to the development of algebraic thinking.[2] All of these curricula seem to emphasize the importance of relationships between quantities, even if the nature of the relationships that are highlighted varies from one to another – with functional relationships being an essential element of some but not others. Additional commonalities across the curricula include a focus on generalization, justification, problem solving, modeling, and noticing structure – with the kind of structure that is articulated differing from one to the other. The kinds of activities that are shared are indeed the very ones that characterize the global, meta-level activities of the model of algebraic activity developed by Kieran (1996). There is thus an opportunity here to have a shared framework for considering algebraic thinking in the early grades – shared not only in the sense, "among the early grades curricula," but also in the sense, "between early grades and later grades."

A Framework for Considering Algebraic Thinking in the Early Grades

In the first section of this paper, I presented a model of algebraic activity that consisted of three types of activity: generational, transformational, and global meta-level (Kieran, 1996). The description of the global meta-level activity was as follows:

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[2] There are a few dangers involved in making general statements about any of these curricula, based on the restricted amount of information contained in the papers in this special issue. Any misinterpretations that occur are however my responsibility.

> Lastly, there are the global, meta-level, mathematical activities. These are the activities for which algebra is used as a tool but which are not exclusive to algebra. They include problem solving, modeling, noticing structure, studying change, generalizing, analyzing relationships, justifying, proving, and predicting – activities that could be engaged in without using any algebra at all. In fact, they suggest more general mathematical processes and activity. However, attempting to divorce these meta-level activities from algebra removes any context or need that one might have for using algebra. Indeed, the global meta-level activities are essential to the other activities of algebra, in particular, to the meaning-building generational activities; otherwise all sense of purpose is lost.

The fact that these activities can be engaged in without using the letter-symbolic, and that they can be further elaborated at any time so as to encompass the letter-symbolic, makes them ideal vehicles for conceptualizing a non-symbolic or pre-symbolic approach to algebraic thinking in the primary grades.

It was perhaps in the paper describing the Korean curriculum that the potential link between algebraic thinking in the early grades and the global meta-level type of algebraic activity, as set forth in the Kieran model, was most obvious. In that paper, Lew made a distinction between algebra as a set of knowledge and techniques (components that were similarly described in the model above as the generational and transformational activities of algebra) and algebra as a way of thinking. As with the global meta-level activities, the ways of thinking articulated by Lew were clearly admitted to be mathematical rather than algebraic (e.g., generalization, abstraction, analytic thinking, modeling, etc), and were argued to be crucial for success in algebra. Lew thus equated the development of algebraic thinking at the elementary level with the development of these mathematical ways of thinking.

By viewing the global meta-level activities of algebra as essential not only for meaning-building in algebra, but also for developing ways of thinking that are crucial for success in algebra, it becomes possible for us to have a vision of algebraic thinking at the early grades that is completely compatible with certain current perspectives on algebraic activity at the later grades. The global meta-level activities of algebra can then be considered not only as part of letter-symbolic algebraic activity but also as precursors to generational and transformational activities to be engaged in later on. The advantage of incorporating a framework for algebraic thinking in the early grades within this existing model is that it bridges a disconnect that has gone on for too long between efforts at introducing algebraic thinking in the early grades and the large body of algebra research that exists with respect to algebra learning and thinking with older students (12 to 13 years of age and beyond).

The three-part model of algebraic activity (Kieran, 1996) that comprises the generational, the transformational, and the global meta-level is a model that has been applied successfully by others in a variety of contexts. For example, Sutherland (1997) used it as a framework for the study she chaired that was conducted under the auspices of the Joint Mathematical Council and The Royal Society on the state of algebra education in the United Kingdom. Brown and Coles (1999) found this model to be "a useful way of describing algebraic activity" (p. 153) in their research involving 11, 15, and 18 year olds. Other researchers who have used this model as a theoretical perspective include the team of Ainley, Bills, and Wilson (Ainley, Wilson, & Bills, 2003; Wilson, Ainley, & Bills, 2003) – in their Purposeful Algebraic Activity Project on the development of algebraic activity in pupils in the early years of secondary schooling. Thus, the framework being proposed for thinking about "algebraic thinking in the early grades of elementary school" is one that is shared by the existing school algebra research community.

To conclude, then, let me offer the following definition for algebraic thinking in the early grades, a definition that is based on the global meta-level activity of algebra as described by Kieran (1996) in her model of the three main activities of school algebra:

> Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting.

References

Ainley, J., Wilson, K., & Bills, L. (2003). Generalising the context and generalising the calculation. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PME-NA* (Vol. 2, pp. 9-16). Honolulu: Center for Research and Development Group, University of Hawaii.

Bednarz, N., Kieran, C., & Lee, L. (Eds.). (1996). *Approaches to algebra: Perspectives for research and teaching*. Dordrecht, The Netherlands: Kluwer.

Bell, A. (1995). Purpose in school algebra. In C. Kieran (Ed.), New perspectives on school algebra: Papers and discussions of the ICME-7 Algebra Working Group (special issue). *Journal of Mathematical Behavior, 14*, 41-73.

Brown, L., & Coles, A. (1999). Needing to use algebra – A case study. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 153-160). Haifa, Israel: PME Program Committee.

Fey, J. T., & Heid, M. K. (1991). *Computer-intensive algebra*. College Park: University of Maryland; University Park: The Pennsylvania State University.

Heid, M. K. (1996). A technology-intensive functional approach to the emergence of algebraic thinking. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 239-255). Dordrecht, The Netherlands: Kluwer.

Kaput, J. J. (1995). A research base supporting long term algebra reform? In D. T. Owens, M. K. Reed, & G. M. Millsaps (Eds.), *Proceedings of the 17th Annual Meeting of PME-NA* (Vol. 1, pp. 71-94). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *8th International Congress on Mathematical Education: Selected lectures* (pp. 271-290). Seville, Spain: S.A.E.M. Thales.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Lee, L. (1997). *Algebraic understanding: The search for a model in the mathematics education community*. Unpublished doctoral dissertation. Université du Québec à Montréal.

Lee, L., & Wheeler, D. (1987). *Algebraic thinking in high school students: Their conceptions of generalisation and justification* (research report). Montreal, QC: Concordia University, Mathematics Department.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Dordrecht, The Netherlands: Kluwer.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. (1998). *The nature and role of algebra in the K-14 curriculum.* Washington: DC: National Academy Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Romberg, T. A., Fennema, E., & Carpenter, T. P. (Eds.). (1993). *Integrating research on the graphical representation of functions*. Hillsdale, NJ: Erlbaum.

Schwartz, J., & Yerushalmy, M. (1992). Getting students to function in and with algebra. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (MAA Notes, Vol. 25, pp. 261-289). Washington, DC: Mathematical Association of America.

Smith, E. (2003). Stasis and change: Integrating patterns, functions, and algebra throughout the K-12 curriculum. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 136-150). Reston, VA: National Council of Teachers of Mathematics.

Sutherland, R. (1997). *Teaching and learning algebra pre-19* (report of a Royal Society / JMC Working Group). London: The Royal Society.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variable. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (1988 Yearbook of the National Council of Teachers of Mathematics, pp. 8-19). Reston, VA: NCTM.

Wagner, S., & Kieran, C. (1989). *Research issues in the learning and teaching of algebra*. Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Erlbaum.

Wilson, K., Ainley, J., & Bills, L. (2003). Comparing competence in transformational and generational algebraic activities. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PME-NA* (Vol. 4, pp. 427-434). Honolulu: Center for Research and Development Group, University of Hawaii.

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